IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Geometric formulation of mechanical systems with one-sided constraints

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1990 J. Phys. A: Math. Gen. 23 2801

(http://iopscience.iop.org/0305-4470/23/13/019)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 01/06/2010 at 08:37

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **23** (1990) 2801-2813. Printed in the UK

Geometric formulation of mechanical systems with one-sided constraints

Ernesto A Lacomba† and W M Tulczyjew‡

† Departamento de Matemáticas, Universidad Autónoma Metropolitana Iztapalapa, Apdo Postal 55-534, CP 09340 México, D F, México
‡ Dipartimento di Matematica e Fisica, Università di Camerino, I-62032 Camerino, Italy

Received 2 October 1989

Abstract. A geometric formulation of analytical mechanics of systems with holonomic and holonomic one-sided constraints is proposed. This is an extension of work by Tulczyjew in which non-holonomic constraints were also considered, but not the one-sided case. The introduction of Lagrange multipliers to give explicit equations of motion was not considered by Tulczyjew either. This allows us to study problems of motion of particles hitting moving walls. We verify that the conditions of collision are satisfied.

1. Geometric structures

Let M be a differentiable manifold. We introduce an equivalence relation in the set $C^\infty(M, \mathbb{R})$ of differentiable curves in M. Two curves $\gamma:\mathbb{R}\to M$ and $\gamma':\mathbb{R}\to M$ are equivalent if $f\circ\gamma'(0)=f\circ\gamma(0)$ and $\mathrm{D}(f\circ\gamma')(0)=\mathrm{D}(f\circ\gamma)(0)$ for each differentiable function f on M. The set of equivalence classes of curves is denoted TM and is called the *tangent bundle* of M. The bundle projection $\tau_M : TM\to M$ is defined by $\tau_M(t_0\gamma)=\gamma(0)$, where $t_0\gamma$ denotes the equivalence class of γ at $\gamma(0)$. If this is done for any value of the parameter and not just at zero, we see that the curve γ lifts to its curve of tangent vectors $t\gamma:\mathbb{R}\to TM$, so that in particular $t_0\gamma=t\gamma(0)$. The *cotangent bundle* T^*M of a manifold M is a vector bundle dual to the tangent bundle TM. Its bundle projection is denoted by $\pi_M : T^*M\to M$.

If functions $(q^\varkappa)$ $(\varkappa=1,2,\ldots,n)$ form a coordinate system in a neighbourhood $U\subset M$, we denote by $(q^\varkappa, \dot q^\lambda)$, $(\varkappa,\lambda=1,2,\ldots,n)$ the corresponding coordinate system in $TU=\tau_M^{-1}(U)$, and by $(q^\varkappa, p_\lambda)$, $(\varkappa,\lambda=1,2,\ldots,n)$ the corresponding coordinates in $T^*U=\pi_M^{-1}(U)$. The coordinates p_λ are denoted by f_λ if the elements of T^*M are generalised forces, or by Δp_λ if they are, rather, impulse jumps at collisions. If C is a submanifold of M, we consider its tangent bundle $TC\subset TM$. For each point of $C\subset M$ there exists a neighbourhood U such that the intersection of C with U is described by equations $C^A(q^\varkappa(q))=0$, where C^A, $(A=1,\ldots,m)$ are functions on $\mathbb{R}^n$. In section 3 we will deal with more general subsets C of M which are submanifolds with one or several boundaries. They are locally described by some equations $C^A(q^\varkappa(q))=0$ $(A=1,\ldots,m)$ as well as some inequalities $C^B(q^\varkappa(q))\geqslant 0$ $(B=1,\ldots,s)$, where C^A and C^B are functions on $\mathbb{R}^n$. If we take local coordinates $(q^\varkappa, \dot q^\lambda)$ $(\varkappa,\lambda=1,\ldots,n)$ in TM, the tangent bundle $TC\subset TM$ is defined locally by conditions $C^A(q^\varkappa(q))=0$, $(\partial C^A/\partial q^\varkappa)\dot q^\varkappa=0$ $(A=1,\ldots,m)$ and either $C^B(q^\varkappa(q))>0$ or $C^B(q^\varkappa(q))=0$ and $(\partial C^B/\partial q^\varkappa)\dot q^\varkappa\geqslant 0$ for each $B=1,\ldots,s$.

0305-4470/90/132801+13\$03.50 © 1990 IOP Publishing Ltd

For each differentiable mapping $\alpha : M \to M'$ one has a differentiable mapping $T\alpha : TM \to TM'$, its tangent mapping. For each differentiable function $g : M \to \mathbb{R}$ we define a function $d_T g : TM \to R$ by $d_T g(t_0\gamma) = \mathrm{D}(g \circ \gamma)(0)$.

Consider coordinates $(q^\varkappa)$ $(\varkappa = 1, 2, \ldots, n)$ in a neighbourhood U of M and coordinates $(q^\varkappa, \dot{q}^\lambda)$ $(\varkappa, \lambda = 1, 2, \ldots, n)$ in $TU = \tau_M^{-1}(U)$, which in turn induce coordinates $(q^\varkappa, \dot{q}^\lambda, \delta q^\mu, \delta\dot{q}^\nu)$ $(\varkappa, \lambda, \mu, \nu = 1, 2, \ldots, n)$ in $TTU = \tau_{TM}^{-1}(\tau_M^{-1}(U))$. If $\chi : \mathbb{R}^2 \to M$ is a differentiable mapping, we denote by $\chi_s : \mathbb{R} \to M$ the curve obtained by fixing $s \in \mathbb{R}$. Then $t_0\chi_s$ is a curve in TM defined by $t_0\chi(s) = t_0\chi_s$. Hence, $t_0t_0\chi$ is an element of TTM and all its elements can be represented this way. Then in terms of this representation we define a mapping $\varkappa_M : TTM \to TTM$ by $\varkappa_M(t_0t_0\chi) = t_0t_0\tilde{\chi}$ where $\tilde{\chi} : \mathbb{R}^2 \to M$ is defined by $\tilde{\chi}(s, t) = \chi(t, s)$. The mapping $\varkappa_M$ is called the *natural involution* in TTM and relations $\varkappa_M \circ \varkappa_M = 1_{TTM}$, $\tau_{TM} \circ \varkappa_M = T\tau_M$ and $T\tau_M \circ \varkappa_M = \tau_{TM}$ are easily verified. The local characterisation of $\varkappa_M$ is provided by relations $q^\varkappa \circ \varkappa_M = q^\varkappa$, $\dot{q}^\lambda \circ \varkappa_M = \delta q^\lambda$, $\delta q^\mu \circ \varkappa_M = \dot{q}^\mu$, $\delta\dot{q}^\nu \circ \varkappa_M = \delta\dot{q}^\nu$.

Let $TM \times_M T^*M$ denote the fibre product of the fibrations $\tau_M : TM \to M$ and $\pi_M : T^*M \to M$. Elements of $TM \times_M T^*M$ are pairs $(v, p) \in TM \times T^*M$ such that $\tau_M(v) = \pi_M(p)$. Since TM and T^*M are dual bundles over M, this induces the *canonical pairing* as a function $\langle\, , \rangle : TM \times_M T^*M \to \mathbb{R}$, by evaluation $\langle v, p\rangle \in \mathbb{R}$. Related to this we have the *canonical* symplectic 2-form ω_M on the manifold T^*M.

A curve $\zeta : \mathbb{R} \to TM \times_M T^*M$ is a pair of curves $\xi : \mathbb{R} \to TM$ and $\eta : \mathbb{R} \to T^*M$ such that $\tau_M \cdot \xi = \pi_M \cdot \eta$. The tangent vector $t_0\zeta$ can be identified with the pair (w, u) of vectors $w = t_0\xi \in TTM$ and $u = t_0\eta \in TT^*M$ satisfying $T\tau_M(w) = T\pi_M(u)$. It follows that the tangent bundle $T(TM \times_M T^*M)$ can be identified with the fibre product $TTM \times_{TM} TT^*M$ of fibrations $T\tau_M : TTM \to TM$ and $T\pi_M : TT^*M \to TM$. Then the mapping $d_T\langle\, , \rangle : T(TM \times_M T^*M) \to \mathbb{R}$ becomes a function on $TTM \times_{TM} TT^*M$. Taking coordinates $(q^\varkappa)$ $(\varkappa = 1, 2, \ldots, n)$ in an open neighbourhood U of M induces coordinates $(q^\varkappa, \delta q^\lambda, \dot{q}^\mu, \delta\dot{q}^\nu)$ $(\varkappa, \lambda, \mu, \nu = 1, 2, \ldots, n)$ in TTU and coordinates $(q^\varkappa, p_\rho, \dot{q}^\mu, \dot{p}_\sigma)$ $(\varkappa, \rho, \mu, \sigma = 1, 2, \ldots, n)$ in TT^*U. The fibre product $TTU \times_{TU} TT^*U$ is the set of pairs (w, u) in $TTU \times TT^*U$ satisfying $q^\varkappa(w) = q^\varkappa(u)$ and $\dot{q}^\mu(w) = \dot{q}^\mu(u)$, while $d_T\langle\, , \rangle$ is described locally by $d_T\langle w, u\rangle = \dot{p}_\varkappa(u)\delta q^\varkappa(w) + p_\varkappa(u)\delta\dot{q}^\varkappa(w)$.

The reader is referred to Tulczyjew (1986) for further details on the geometrical constructions.

2. Dynamics with holonomic constraints

Let M be the *configuration manifold* of a mechanical system with holonomic constraints and external forces. The motion is a curve in $T^*M \times_M T^*M$, i.e. a pair (η, φ) of curves in T^*M such that $\pi_M \circ \eta = \pi_M \circ \varphi$. Here $T^*M \times_M T^*M$ is interpreted as the fibre product over M of the *momentum bundle* with the *force bundle*. The equation of motion is the condition that the image of the curve $(t\eta, \varphi)$ is contained in a submanifold $\dot{D}$ of the fibre product of the fibrations $\pi_M \circ \tau_{T^*M} : TT^*M \to M$ and $\pi_M : T^*M \to M$. The submanifold $\dot{D}$ is usually defined by a variational principle of the form

$$\dot{D} = \{(w, f) \in TT^*M \times_M T^*M;\ T\pi_M(w) \in TC,\ d_T\langle \varkappa_M(v), w\rangle - \langle T\tau_M(v), f\rangle$$
$$= \langle v, dL\rangle \text{ if } T\tau_M(v) \in TC,\ \tau_{TM}(v) = T\pi_M(w)\}$$

where $L : TM \to \mathbb{R}$ is the Lagrangian and C is a submanifold of M. For a system without constraints, $C = M$.

We consider first the time-independent case where M contains only position coordinates. A local coordinate system $(q^\varkappa)$ $(\varkappa = 1, \ldots, n)$ in M induces local coordinates $(q^\varkappa, f_\lambda)$ $(\varkappa, \lambda = 1, \ldots, n)$ in the force bundle T^*M. In TT^*M and TTM we have induced coordinates $(q^\varkappa, p_\lambda, \dot{q}^\mu, \dot{p}_\nu)$ and $(q^\varkappa, \dot{q}^\lambda, \delta q^\mu, \delta\dot{q}^\nu)$, $(\varkappa, \lambda, \mu, \nu = 1, \ldots, n)$. The submanifold C of M is described locally by $C^A(q^\varkappa) = 0$, $(A = 1, \ldots, m)$. The local form of the variational principle defining $\dot{D}$ is

$$\begin{aligned} &C^A(q^\varkappa) = 0 \qquad \frac{\partial C^A}{\partial q^\varkappa}\dot{q}^\varkappa = 0 \\ &(\dot{p}_\varkappa - f_\varkappa)\delta q^\varkappa + p_\varkappa \delta\dot{q}^\varkappa = \frac{\partial L}{\partial q^\varkappa}\delta q^\varkappa + \frac{\partial L}{\partial \dot{q}^\varkappa}\delta\dot{q}^\varkappa \end{aligned} \tag{1}$$

for all $\delta q^\varkappa$, $\delta\dot{q}^\lambda$ such that $(\partial C^A/\partial q^\varkappa)\delta q^\varkappa = 0$, $(A = 1, \ldots, m)$, where $q^\varkappa = q^\varkappa(f) = q^\varkappa(v) = q^\varkappa(w)$, $\dot{q}^\varkappa = \dot{q}^\varkappa(v) = \dot{q}^\varkappa(w)$, $f_\varkappa = f_\varkappa(f)$, $p_\varkappa = p_\varkappa(w)$, $\dot{p}_\varkappa = \dot{p}_\varkappa(w)$, $\delta q^\varkappa = \delta q^\varkappa(v)$ and $\delta\dot{q}^\varkappa = \delta\dot{q}^\varkappa(v)$.

If it were not for the forces $f_\varkappa$, we could apply here the standard Lagrange multiplier procedure in the calculus of variations. The more general statement we need here depends only on linear algebra arguments at the derivatives level. Assume that the $m \times n$ matrix $(C^A_\varkappa)$ has maximal rank m. Let $\boldsymbol{C}^A = (C^A_1, \ldots, C^A_n) \in \mathbb{R}^n$ for any index $A = 1, 2, \ldots, m$ be the row vectors of the matrix and let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$.

Proposition 1. If $a_\varkappa \delta q^\varkappa = 0$ for any $\delta q^\varkappa$ satisfying $C^A_\varkappa \delta q^\varkappa = 0$, then there exist Lagrange multipliers $\lambda_A \in \mathbb{R}$ such that

$$a_\varkappa \delta q^\varkappa = \lambda_A C^A_\varkappa \delta q^\varkappa$$

for any $\delta q^\varkappa$, and conversely.

Proof. Denote by $sp\{\boldsymbol{a}\}$ and $sp\{\boldsymbol{C}^1, \ldots, \boldsymbol{C}^m\}$ the vector subspaces spanned by the given vectors. By hypothesis, the corresponding orthogonal complements satisfy $sp\{\boldsymbol{C}^1, \ldots, \boldsymbol{C}^m\}^\perp \subset sp\{\boldsymbol{a}\}^\perp$. Hence $\boldsymbol{a} \in sp\{\boldsymbol{C}^1, \ldots, \boldsymbol{C}^m\}$, as required. The converse is trivial. □

Equations (1) may be rewritten as follows if we introduce Lagrange multipliers λ_A $(A = 1, \ldots, m)$ to take into account the condition $(\partial C^A/\partial q^\varkappa)\,\delta q^\varkappa = 0$. Using the above proposition with $C^A_\varkappa = \partial C^A/\partial q^\varkappa$, we get

$$(\dot{p}_\varkappa - f_\varkappa)\delta q^\varkappa + p_\varkappa \delta\dot{q}^\varkappa = \frac{\partial L}{\partial q^\varkappa}\delta q^\varkappa + \frac{\partial L}{\partial \dot{q}^\varkappa}\delta\dot{q}^\varkappa + \lambda_A \frac{\partial C^A}{\partial q^\varkappa}\delta q^\varkappa$$

for any $\delta q^\varkappa$, $\delta\dot{q}^\varkappa$. From here, we immediately get the following differential equations for the dynamics:

$$\dot{p}_\varkappa = \frac{\partial L}{\partial q^\varkappa} + f_\varkappa + \lambda_A \frac{\partial C^A}{\partial q^\varkappa} \qquad p_\varkappa = \frac{\partial L}{\partial \dot{q}^\varkappa}. \tag{2}$$

Let us now consider the homogeneous time-dependent case. We take position and time coordinates $(q^\varkappa, t)$, $(\varkappa = 1, \ldots, n)$ in M, which induce local coordinates $(q^\varkappa, t, g_\lambda, j)$ $(\varkappa, \lambda = 1, \ldots, n)$ in the force bundle T^*M. They also induce canonical coordinates $(q^\varkappa, t, p_\lambda, u, q'^\mu, t', p'_\nu, u')$ $(\varkappa, \lambda, \mu, \nu = 1, \ldots, n)$ in TT^*M where the prime denotes derivatives with respect to a new parameter, as well as coordinates

$(q^\varkappa, t, q'^\lambda, t', \delta q^\mu, \delta t, \delta q'^\nu, \delta t')$ $(\varkappa, \lambda, \mu, \nu = 1, \ldots, n)$ in TTM. The submanifold C of M is described by $C^A(q^\varkappa, t) = 0$ $(A = 1, \ldots, m)$ and it represents a time-dependent holonomic constraint. The local form of the variational principle defining $\dot{D}$ is now written as

$$C^A(q^\varkappa, t) = 0 \qquad \frac{\partial C^A}{\partial q^\varkappa} q'^\varkappa + \frac{\partial C^A}{\partial t} t' = 0$$
$$(p'_\varkappa - g_\varkappa)\delta q^\varkappa + (u' - j)\delta t + p_\varkappa \delta q'^\varkappa + u\delta t' = \delta L(q^\varkappa, t, q'^\lambda, t') \tag{3}$$

for all $\delta q^\varkappa$, δt, $\delta q'^\lambda$, $\delta t'$ such that $(\partial C^A/\partial q^\varkappa)\delta q^\varkappa + (\partial C^A/\partial t)\delta t = 0$ $(A = 1, \ldots, m)$, where besides the corresponding conditions on the coordinates as in the other case, we have $t = t(f) = t(v) = t(w)$, $q'^\varkappa = q'^\varkappa(v) = q'^\varkappa(w)$, $q_\varkappa = g_\varkappa(f)$, $j = j(f)$, $p'_\varkappa = p'_\varkappa(w)$, $t' = t'(v) = t'(w)$, $\delta t = \delta t(v)$ and $\delta q'^\varkappa = \delta q'^\varkappa(v)$. Also, $L(q^\varkappa, t, q'^\varkappa, t')$ is homogeneous in the velocities.

We change to more convenient non-canonical coordinates $(q^\varkappa, t, p_\lambda, e, \dot{q}^\mu, t', \dot{p}_\nu, \dot{e})$ on TT^*M with time derivatives, except that derivative of time with respect to the new parameter does still appear. It is defined by the equations $e = -u$, $t'\dot{q}^\mu = q'^\mu$, $t'\dot{p}_\nu = p'_\nu$ and $t'\dot{e} = -u'$ where, of course, $t' \neq 0$. Since a change of coordinates from $(q^\varkappa, t, q'^\lambda, t')$ to $(q^\varkappa, t, \dot{q}^\lambda, t')$, $(\varkappa, \lambda = 1, \ldots, n)$ in TM induces one in TTM by taking derivatives, this means that we can write $\delta q'^\varkappa = t'\delta\dot{q}^\varkappa + \dot{q}^\varkappa \delta t'$. By also renaming the coordinates $g_\varkappa = t'f_\varkappa$, $j = -t'h$ in the force bundle T^*M, we see that (3) may be rewritten by introducing Lagrange multipliers from proposition 1 again, getting

$$\begin{aligned} &t'[(\dot{p}_\varkappa - f_\varkappa)\delta q^\varkappa - (\dot{e} - h)\delta t + p_\varkappa \delta\dot{q}^\varkappa] + (p_\varkappa \dot{q}^\varkappa - e)\delta t' \\ &\qquad = t'\frac{\partial \bar{L}}{\partial q^\varkappa}\delta q^\varkappa + t'\frac{\partial \bar{L}}{\partial \dot{q}^\varkappa}\delta\dot{q}^\varkappa + t'\frac{\partial \bar{L}}{\partial t}\delta t + \bar{L}\delta t' + t'\lambda_A\left(\frac{\partial C^A}{\partial q^\varkappa}\delta q^\varkappa + \frac{\partial C^A}{\partial q^\varkappa}\delta t\right) \end{aligned}$$

for any $\delta q^\varkappa$, $\delta q'^\lambda$, δt, $\delta t'$. Here $\bar{L}(q^\varkappa, t, \dot{q}^\lambda) = L(q^\varkappa, t, \dot{q}^\lambda, 1)$ is the non-homogeneous Lagrangian, i.e. $L(q^\varkappa, t, q'^\lambda, t') = t'\bar{L}(q^\varkappa, t, q'^\lambda)$. As before, we directly obtain the differential equations for the dynamics as

$$\dot{p}_\varkappa = \frac{\partial \bar{L}}{\partial q^\varkappa} + f_\varkappa + \lambda_A \frac{\partial C^A}{\partial q^\varkappa} \qquad p_\varkappa = \frac{\partial \bar{L}}{\partial \dot{q}^\varkappa}$$
$$\dot{e} - h = -\frac{\partial \bar{L}}{\partial t} - \lambda_A \frac{\partial C^A}{\partial t} \qquad p_\varkappa \dot{q}^\varkappa - e = \bar{L} \tag{4}$$

the variable e is the energy of the system, while h is interpreted as the power of the external forces $f_\varkappa$.

3. Dynamics with one-sided constraints

In this section M will be the configuration manifold of a mechanical system with holonomic constraints, one-sided conditions and external forces. This includes the dynamics of particles bouncing elastically against walls (which may eventually be in motion as well), or even with motion constrained to a submanifold and bouncing against a (moving) wall on it. The constraint subsets C in M will be manifolds with one or several boundaries, as described in the introduction.

We will restrict ourselves to the homogeneous time-dependent case. Conservation of energy is not obtained in the time-independent case.

The motion is a curve in $T^*M \times_M T^*M$, i.e. a pair (η, φ) of curves in T^*M such that $\pi_M \circ \eta = \pi_M \circ \varphi = \gamma$. The projection curve γ is continuous and differentiable from above. The curves η and φ are not in general continuous, but possess lateral limits and are differentiable from above. The *jumping curve* $\Delta\eta$ of η is defined as the difference between its lateral limits. The equation of motion is the condition that the image of the curve $(t\eta, \Delta\eta, \varphi)$ is contained in a subset $\dot{D}$ of the fibre product $TT^*M \times_M T^*M \times_M T^*M$. This subset is usually defined by a variational principle of the form

$$\dot{D} = \{(w, r, f) \in TT^*M \times_M T^*M \times_M T^*M;\ T\pi_M(w) \in TC,\ d_T\langle \varkappa_M(v), w\rangle - \langle T\tau_M(v), f\rangle$$
$$\geqslant \langle v, dL\rangle \text{ and } \langle T\tau_M(v), r\rangle \geqslant 0 \text{ if } T\tau_M(v) \in TC,\ \tau_{TM}(v) = T\pi_M(w)\}$$

where $L: TM \to \mathbb{R}$ is the homogeneous Lagrangian and C is the smooth constraint subset of M, with tangent bundle TC.

We take coordinates here which are an extension of those considered in the homogeneous holonomic case. We consider position and time coordinates $(q^\varkappa, t)$, $(\varkappa = 1, \ldots, n)$ in M, inducing local coordinates $(q^\varkappa, t, \Delta p_\lambda, \Delta u)$ in the *jump bundle* T^*M and $(q^\varkappa, t, g_\lambda, j)$ $(\varkappa, \lambda = 1, \ldots, n)$ in the force bundle T^*M. It also induces *canonical* coordinates $(q^\varkappa, t, p_\lambda, u, q'^\mu, t', p'_\nu, u')$ in TT^*M as well as coordinates $(q^\varkappa, t, q'^\lambda, t', \delta q^\mu, \delta t, \delta q'^\nu, \delta t')$, $(\varkappa, \lambda, \mu, \nu = 1, \ldots, n)$ in TTM. The subset C of M is described locally by $C^A(q^\varkappa, t) = 0$ $(A = 1, \ldots, m)$ and $C^B(q^\varkappa, t) \geqslant 0$ $(B = m+1, \ldots, s)$. The tangent bundle TC can be described correspondingly as we show below, according to the local coordinates in TM being induced by the projections $T\tau_M$ or $T\pi_M$. The local form of the variational principle defining $\dot{D}$ is written as

$$C^A(q^\varkappa, t) = 0 \qquad \frac{\partial C^A}{\partial q^\varkappa} q'^\varkappa + \frac{\partial C^A}{\partial t} t' = 0 \ (A = 1, \ldots, m)$$

and either

$$C^B(q^\varkappa, t) > 0$$

or

$$C^B(q^\varkappa, t) = 0 \tag{5}$$

and

$$\frac{\partial C^B}{\partial q^\varkappa} q'^\varkappa + \frac{\partial C^B}{\partial t} t' \geqslant 0 \ (B = m+1, \ldots, s)$$

$$(p'_\varkappa - g_\varkappa)\delta q^\varkappa + (u' - j)\delta t + p_\varkappa \delta q'^\varkappa + u\delta t'^\varkappa \geqslant \delta L(q^\varkappa, t, q'^\lambda, t') \tag{6}$$

and $\Delta p_\varkappa \delta q^\varkappa + \Delta u \delta t \geqslant 0$ for all $\delta q^\varkappa$, δt, $\delta q'^\lambda$, $\delta t'$ such that (5) is satisfied with $\delta q^\varkappa$, δt, replacing $q'^\varkappa$ and t.

In order to get the equations of motion, we need the following generalisation of the Lagrange multipliers statement in proposition 1, which includes one-sided constraints.

We remark that in general the boundary defined by the one-sided constraints in (5) has several components where some or all of the defining functions C^B $(B = m + 1, \ldots, s)$ annihilate. For each boundary component we may relabel the indices B so that it can be defined by $C^A=0$ and $C^B=0$ for all $A=1,\ldots,m$ and $B=m+1,\ldots,r$ with $m+1\leqslant r\leqslant s$. A reasonable assumption on the constraints is that the matrix

$$\left(\frac{\partial C^A}{\partial q^\varkappa}\right)$$

$(A=1,\ldots,r)$, $(\varkappa=1,\ldots,n+1)$ where $q^{n+1}=t$, has maximal rank r on any such boundary component. The interior set is defined by the condition that all the constraint functions are positive. More generally, in the proposition to follow, we deal with an $r\times(n+1)$ matrix $(C^A_\varkappa)$ of rank r.

Proposition 2. The following two conditions are equivalent.

(a) $a_\varkappa\delta q^\varkappa\geqslant 0$ for $\delta q^\varkappa$ satisfying $C^A_\varkappa\delta q^\varkappa=0$ $(A=1,2,\ldots,m)$, $C^B_\varkappa\delta q^\varkappa\geqslant 0$, $(B=m+1,\ldots,r\leqslant s)$.

(b) $a_\varkappa=\lambda_A C^A_\varkappa+\lambda_B C^B_\varkappa$ for some $\lambda_A\in\mathbb{R}$, $(A=1,2,\ldots,m)$ $\lambda_B\geqslant 0$, $(B=m+1,\ldots,r)$.

Proof. Clearly (b) implies (a). We will now prove that (a) implies (b). Part of the proof will be a reduction to proposition 1. If we denote $\boldsymbol{\delta q}=(\delta q^1,\ldots,\delta q^{n+1})\in\mathbb{R}^{n+1}$, each subset $(\boldsymbol{\delta q}:C^B_\varkappa\delta q^\varkappa\geqslant 0\}$ as well as $\{\boldsymbol{\delta q}:a_\varkappa\delta q^\varkappa\geqslant 0)$ is a closed half-space in $\mathbb{R}^{n+1}$.

We remark that any non-trivial vector subspace contained in a closed half-space has to be contained in its boundary hyperplane. Hence, if $\boldsymbol{\delta q}$ satisfies the equalities $C^A_\varkappa\delta q^\varkappa=0$, $C^B_\varkappa\delta q^\varkappa=0$ for any $A=1,\ldots,m$ and $B=m+1,\ldots,r$ then the equality $a_\varkappa\delta q^\varkappa=0$ holds. Hence, we can apply proposition 1 to conclude that for some coefficients $\lambda_A\in\mathbb{R}$ and $\lambda_B\in\mathbb{R}$ we have

$$a_\varkappa=\lambda_A C^A_\varkappa+\lambda_B C^B_\varkappa.$$

Now,

$$a_\varkappa\delta q^\varkappa=\lambda_A C^A_\varkappa\delta q^\varkappa+\lambda_B C^B_\varkappa\delta q^\varkappa=\lambda_B(C^B_\varkappa\delta q^\varkappa)$$

must be non-negative whenever $C^A_\varkappa\delta q^\varkappa=0$, $C^B_\varkappa\delta q^\varkappa\geqslant 0$. Since the vectors $\{\boldsymbol{C}^1,\ldots,\boldsymbol{C}^m,\boldsymbol{C}^{m+1},\ldots,\boldsymbol{C}^r\}$ are linearly independent in $\mathbb{R}^{n+1}$, we can choose $\boldsymbol{\delta q}\in\mathbb{R}^{n+1}$ orthogonal to all of them but one, say $\boldsymbol{C}^B$ for $m+1\leqslant B\leqslant r$. Hence $\lambda_B\geqslant 0$. □

In order to apply this result to the two variational inequalities (6) with constraints (5), we use Lagrange multipliers λ_A, λ_B and μ_A, μ_B. With the more convenient non-canonical coordinates $(q^\varkappa, t, p_\lambda, e, \dot q^\mu, t', \dot p_\nu, \dot e)$ on TT^*M, where $e=-u$, $t'\dot q^\mu=q'^\mu$, $t'\dot p_\nu=p'_\nu$ and $t'\dot e=-u'$ for a non-homogeneous Lagrangian $\bar L(q^\varkappa, t, \dot q^\lambda)$ such that $L=t'\bar L$ as in section 2, we see that (6) is rewritten as

$$\begin{aligned}t'[(\dot p_\varkappa-f_\varkappa)\delta q^\varkappa-(\dot e-h)\delta t+p_\varkappa\delta\dot q^\varkappa]+(p_\varkappa\dot q^\varkappa-e)\delta t'\\ =t'\frac{\partial\bar L}{\partial q^\varkappa}\delta q^\varkappa+t'\frac{\partial\bar L}{\partial\dot q^\varkappa}\delta\dot q^\varkappa+t'\frac{\partial\bar L}{\partial t}\delta t+\bar L\delta t'+t'\lambda_A\left(\frac{\partial C^A}{\partial q^\varkappa}\delta q^\varkappa+\frac{\partial C^A}{\partial t}\delta t\right)\\ +t'\lambda_B\left(\frac{\partial C^B}{\partial q^\varkappa}\delta q^\varkappa+\frac{\partial C^B}{\partial t}\delta t\right)\qquad\text{some }\lambda_A\in\mathbb{R},\ \lambda_B\geqslant 0\end{aligned}\tag{7}$$

and

$$\Delta p_\varkappa\delta q^\varkappa-\Delta e\delta t=\mu_A\left(\frac{\partial C^A}{\partial q^\varkappa}\delta q^\varkappa+\frac{\partial C^A}{\partial t}\delta t\right)+\mu_B\left(\frac{\partial C^B}{\partial q^\varkappa}\delta q^\varkappa+\frac{\partial C^B}{\partial t}\delta t\right)$$

for some $\mu_A \in \mathbb{R}$, $\mu_B \geqslant 0$, $(A=1,\ldots,m)$, $(B=m+1,\ldots,s)$ for any $\delta q^\varkappa$, $\delta \dot{q}'^\lambda$, δt, $\delta t'$. Then we get the following equations for the dynamics and the jumps:

$$
\begin{aligned}
&\dot{p}_\varkappa = \frac{\partial \bar{L}}{\partial q^\varkappa} + f_\varkappa + \lambda_A \frac{\partial C^A}{\partial q^\varkappa} + \lambda_B \frac{\partial C^B}{\partial q^\varkappa} \\
&-(\dot{e}-h) = \frac{\partial \bar{L}}{\partial t} + \lambda_A \frac{\partial C^A}{\partial t} + \lambda_B \frac{\partial C^B}{\partial t} \\
&p_\varkappa = \frac{\partial \bar{L}}{\partial \dot{q}^\varkappa} \qquad p_\varkappa \dot{q}^\varkappa - e = \bar{L} \\
&\Delta p_\varkappa = \mu_A \frac{\partial C^A}{\partial q^\varkappa} + \mu_B \frac{\partial C^B}{\partial q^\varkappa} \qquad \Delta e = -\mu_A \frac{\partial C^A}{\partial t} - \mu_B \frac{\partial C^B}{\partial t}.
\end{aligned}
\tag{8}
$$

Notice that the fourth equation gives exactly the transformation from the energy function to the Lagrangian.

In order to check that the momenta and the energy actually have discontinuity jumps only at the boundary defined by the one-sided conditions, we will prove the following two propositions and their corollary. These finer results depend not only on linear algebra considerations, but also on the metric defined by the kinetic energy of our mechanical system.

Proposition 3 (holonomic constraints). If $\Delta p_\varkappa$ $(\varkappa = 1, 2, \ldots, n)$ and Δe are the momenta and energy jumps at some point and $\Delta p_\varkappa \delta q^\varkappa - \Delta e \delta t = 0$ for $\delta q^\varkappa$, δt satisfying $(\partial C^A/\partial q^\varkappa)\delta q^\varkappa + (\partial C^A/\partial t)\delta t = 0$ $(A=1,\ldots,m)$, then $\Delta p_\varkappa = 0$ for any $\varkappa$, and $\Delta e = 0$.

Proof. From proposition 1 we know that

$$\Delta p_\varkappa = \mu_A \frac{\partial C^A}{\partial q^\varkappa} \qquad \Delta e = -\mu_A \frac{\partial C^A}{\partial t} \qquad (\varkappa = 1, 2, \ldots, n). \tag{9}$$

The momenta are related to the velocities by means of the equation

$$p_\varkappa = g_{\varkappa\lambda} \dot{q}^\lambda \tag{10}$$

where $(g_{\varkappa\lambda})$ is the matrix of the Riemannian metric associated with the kinetic energy. Since the velocities must be tangent to the constraint submanifold, they satisfy the conditions

$$\dot{q}^\varkappa \frac{\partial C^A}{\partial q^\varkappa} + \frac{\partial C^A}{\partial t} = 0 \qquad \text{for any index } A. \tag{11}$$

From (10), we obtain the relationship between the jumps as

$$\Delta p_\varkappa = g_{\varkappa\lambda} \Delta \dot{q}^\lambda.$$

Using the fact that the Riemmanian matrix is symmetrical and combining with (9), we get

$$\Delta \dot{q}^\varkappa g_{\varkappa\lambda} = \Delta p_\lambda = \mu_A \frac{\partial C^A}{\partial q^\lambda}$$

or equivalently

$$\Delta \dot{q}^\varkappa = \mu_A \frac{\partial C^A}{\partial q^\lambda} g^{\lambda\varkappa} \tag{12}$$

where $(g^{\lambda\varkappa}) = (g_{\varkappa\lambda})^{-1}$. On the other hand, the velocity jumps $\Delta\dot{q}^{\varkappa}$ must satisfy the linear part of equation (11):

$$\Delta\dot{q}^{\varkappa} \frac{\partial C^D}{\partial q^{\varkappa}} = 0$$

for any $D = 1, \ldots, m$. Combining with (12), we get

$$\mu_A \frac{\partial C^A}{\partial q^{\lambda}} g^{\lambda\varkappa} \frac{\partial C^D}{\partial q^{\varkappa}} = 0. \tag{13}$$

The expression which multiplies the μ_A can be interpreted as a Grammian matrix defined by the inner products of the rows of the Jacobian matrix $(\partial C^A/\partial q^{\varkappa})$. Since the Jacobian matrix is non-singular, the Grammian is also non-singular, so that $\mu_A = 0$ for any $A = 1, \ldots, m$. Hence, (9) implies that $\Delta p_{\varkappa} = 0$ ($\varkappa = 1, 2, \ldots, n$) and $\Delta e = 0$. □

Equation (13) can be written as

$$\Delta p_{\lambda} g^{\lambda\varkappa} \frac{\partial C^D}{\partial q^{\varkappa}} = 0 \tag{14}$$

which we will consider in the proof of proposition 4.

Proposition 4 (holonomic and one-sided constraints).
(a) On the boundary component defined by $C^A = 0$ for $A = 1, \ldots, m$ and $C^B = 0$, $B = m+1, \ldots, r$ with $m+1 \leqslant r \leqslant s$, if we have $\Delta p_{\varkappa}\delta q^{\varkappa} - \Delta e \delta t \geqslant 0$, then

$$\Delta p_{\varkappa} = \mu_A \frac{\partial C^A}{\partial q^{\varkappa}} + \mu_B \frac{\partial C^B}{\partial q^{\varkappa}} \qquad \Delta e = -\mu_A \frac{\partial C^A}{\partial t} - \mu_B \frac{\partial C^B}{\partial t}$$

for $\varkappa = 1, \ldots, n$, with $\mu_A \in \mathbb{R}$ and $\mu_B \geqslant 0$.
(b) Outside any one-sided boundary, the terms containing the C^B disappear and we get $\Delta p_{\varkappa} = 0$ and $\Delta e = 0$.

Proof. As in proposition 3, we have $p_{\varkappa} = g_{\varkappa\lambda}\dot{q}^{\lambda}$.
(a) The constraints on the velocities are written as

$$\dot{q}^{\varkappa} \frac{\partial C^A}{\partial q^{\varkappa}} + \frac{\partial C^A}{\partial t} = 0$$

$$\dot{q}^{\varkappa} \frac{\partial C^B}{\partial q^{\varkappa}} + \frac{\partial C^B}{\partial t} \geqslant 0 \qquad \text{when } C^B = 0 \tag{15}$$

for $A = 1, \ldots, m$, $B = m+1, \ldots, r$. From proposition 2 we can write

$$\Delta p_{\varkappa} = \mu_A \frac{\partial C^A}{\partial q^{\varkappa}} + \mu_B \frac{\partial C^B}{\partial q^{\varkappa}} \qquad \Delta e = -\mu_A \frac{\partial C^A}{\partial t} - \mu_B \frac{\partial C^B}{\partial t} \tag{16}$$

where $\mu_A \in \mathbb{R}$ and $\mu_B \geqslant 0$, ($A = 1, \ldots, m$), ($B = m+1, \ldots, r$), and we will give a procedure to determine the μ_A.
Multiplying (16) by $\delta q^{\varkappa}$, we get

$$\Delta p_{\varkappa}\delta q^{\varkappa} = \mu_A \frac{\partial C^A}{\partial q^{\varkappa}} \delta q^{\varkappa} + \mu_B \frac{\partial C^B}{\partial q^{\varkappa}} \delta q^{\varkappa}. \tag{17}$$

We can not assert in general that all the $\mu_A = 0$, but since the velocity jumps $\Delta \dot{q}^\varkappa$ will have to satisfy only the equality constraints, equations (14) are still valid:

$$\Delta p_\lambda g^{\lambda\varkappa} \frac{\partial C^D}{\partial q^\varkappa} = 0 \qquad \text{for all } D = 1, 2, \ldots, m.$$

Replacing δq^λ by $g^{\lambda\varkappa}(\partial C^D/\partial q^\varkappa)$ in (17), we get

$$\mu_A \frac{\partial C^A}{\partial q^\lambda} g^{\lambda\varkappa} \frac{\partial C^D}{\partial q^\varkappa} + \mu_B \frac{\partial C^B}{\partial q^\lambda} g^{\lambda\varkappa} \frac{\partial C^D}{\partial q^\varkappa} = 0.$$

As remarked in proposition 3, the matrix multiplying the μ_A in the first term is non-singular. So, we can solve for the μ_A in terms of the μ_B.

Part (b) is a direct consequence of proposition 3. □

From propositions 2 and 4 we conclude the following about equations (8).

Corollary. We have for any given index $B = m+1, \ldots, s$ that $\lambda_B \geq 0$ and $\mu_B \geq 0$ on the subset of the boundary $\{C_B = 0\}$ and they are zero elsewhere, while λ_A, $\mu_A \in \mathbb{R}$ and the μ_A $(A = 1, \ldots, m)$ vanish outside any boundary. In particular, off boundaries we have $\Delta p_\varkappa = 0$ for $\varkappa = 1, 2, \ldots, n$ and $\Delta e = 0$.

4. Examples

In this section we will illustrate our formulation in a few examples. We will also verify that the jump conditions for a particle hitting a wall agree with those of an elastic bounce.

We begin with the simple example of a constrained particle moving in the plane, in order to verify how the jumping parameters and multipliers must be zero.

Example 1. Motion in the plane constrained to the x axis. The Lagrangian is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, subject to the holonomic constraint C: $y = 0$. Hence the corresponding infinitesimal conditions are $\dot{y} = 0$ and $\delta y = 0$. We consider the variational equations in the form (7) as if there were any jumps, writing directly the equations corresponding to equations (8):

$$\dot{p}_x = f_x \qquad p_x = m\dot{x} \qquad \dot{e} - h = 0 \qquad \Delta p_x = 0 \qquad \Delta e = 0$$

$$\dot{p}_y = f_y + \lambda \qquad p_y = m\dot{y} \qquad e = \frac{1}{2m}(p_x^2 + p_y^2) \qquad \Delta p_y = \mu.$$

Here λ and μ are Lagrange multipliers. Since $\dot{y} = 0$, we have $\Delta \dot{y} = 0$, so that $\Delta p_y = m\Delta \dot{y} = 0$ and there is no jump in the p_y, i.e. $\mu = 0$. Also, $p_y = m\dot{y} \equiv 0$, and hence the y component of the force acting on the particle is $\dot{p}_y \equiv 0$. The multiplier λ is the constraint force (principle of d'Alembert) needed to maintain the particle on the x axis, which is just $\lambda = -f_y$.

Finally, $h = \dot{e} = (1/m)p_x\dot{p}_x = \dot{x}f_x$ is indeed the power of the external force.

Example 2. Consider now the general case of collision of a free particle of mass m against a moving wall in $\mathbb{R}^n$. If we write $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, etc., we have a constraint of the form $C(q, t) \geq 0$. The variational equation for the jumping now becomes

$$(p'_\varkappa - p_\varkappa)\delta q^\varkappa - (e' - e)\delta t = \lambda\left(\frac{\partial C}{\partial q^\varkappa}\delta q^\varkappa + \frac{\partial C}{\partial t}\delta t\right)$$

for a multiplier $\lambda \geqslant 0$. Hence, if we denote by grad C the gradient of C with respect to the position variables, we get the following vector and scalar equations:

$$p' - p = \lambda \text{ grad } C(q, t) \tag{18}$$

$$e' - e = -\lambda \frac{\partial C}{\partial t}. \tag{18'}$$

Notice that no motion of the wall implies energy conservation.

At any fixed (q, t) we may express the momentum as $p = p^{\perp} + p^{\parallel}$ in terms of its components normal and parallel to the wall, and similarly for p'. Hence $p' - p = (p'^{\perp} - p^{\perp}) + (p'^{\parallel} - p^{\parallel})$, and from (18) we have, of course,

$$p'^{\parallel} - p^{\parallel} = 0 \qquad p'^{\perp} - p^{\perp} = \lambda \text{ grad } C.$$

On the other hand,

$$e = \frac{1}{2m}(|p^{\perp}|^2 + |p^{\parallel}|^2) \qquad e' = \frac{1}{2m}(|p'^{\perp}|^2 + |p'^{\parallel}|^2)$$

so that

$$e' - e = \frac{1}{2m}(|p'^{\perp}|^2 - |p^{\perp}|^2) = \frac{\lambda}{2m}(p'^{\perp} + p^{\perp}) \cdot \text{grad } C. \tag{19}$$

If $C(q, t) > 0$ then $\lambda = 0$ and (19) agrees with (18′). At a collision we have $C(q, t) = 0$ with $\lambda > 0$. If $V \in \mathbb{R}^n$ denotes the velocity of a point remaining always in the wall, by differentiation we get

$$V \cdot \text{grad } C + \frac{\partial C}{\partial t} = 0$$

or equivalently

$$V^{\perp} \cdot \text{grad } C + \frac{\partial C}{\partial t} = 0. \tag{20}$$

In fact, only $V^{\perp}$ is uniquely defined, unless information is known about motion of individual particles of the wall. Combining with (18′) and (19), we conclude as required that

$$V^{\perp} = \frac{1}{2m}(p'^{\perp} + p^{\perp}). \tag{21}$$

We will now verify that this agrees with the computation of velocities just before and after an elastic bouncing in one dimension. Let V and $V' \in \mathbb{R}$ be the velocity of the wall just before and after the collision, while v and $v' \in \mathbb{R}$ are the velocities of the particle just before and after. Then momentum conservation gives

$$V' - V = \frac{m}{M}(v - v')$$

while the quotient of kinetic energy conservation divided by momentum conservation in the above form gives

$$V' + V = v + v'$$

where M is the mass of the wall. Since we assume $M \to \infty$ for the mass of the wall, we get $V' = V$. So $v + v' = 2V$ or

$$\frac{p'+p}{2m} = V$$

in terms of momenta, which is a particular case of (21).

Example 3. Consider the motion of a particle constrained to an arc of circle in the plane. We consider more precisely the arc of the unit circle contained in the first quadrant.

The Lagrangian is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and the constraints are

$$\begin{array}{lll} C^1: & x^2+y^2=1 & \text{so } x\dot{x}+y\dot{y}=0 \text{ and } x\delta x+y\delta y=0 \\ C^2: & x \geqslant 0 & \text{so } \dot{x} \geqslant 0 \text{ if } x=0 \\ C^3: & y \geqslant 0 & \text{so } \dot{y} \geqslant 0 \text{ if } y=0. \end{array}$$

In this case the equations (8) become

$$\begin{array}{lll} \dot{p}_x = \lambda_1 x + \lambda_2 & \dot{e} - h = 0 & p_x = m\dot{x} \\ \Delta p_x = \mu_1 x + \mu_2 & \Delta e = 0 & \dot{p}_y = \lambda_1 y + \lambda_3 \\ e = \dfrac{1}{2m}(p_x^2 + p_y^2) & p_y = m\dot{y} & \Delta p_y = \mu_1 y + \mu_3 \end{array}$$

where $\lambda_1, \mu_1 \in \mathbb{R}$ and the Lagrange multipliers $\lambda_2, \lambda_3, \mu_2, \mu_3$ of the one-sided conditions are non-negative. Since $(\dot{p}_x, \dot{p}_y)$ is the total force acting on the particle, we recognise $\lambda_1(x, y)$ as the d'Alembert force needed to maintain the particle on the arc. Then (λ_2, λ_3) gives the force at the endpoints of the arc making the particle bounce back, and hence it is zero in its interior. At $x = 0$ we have $\lambda_2 > 0$ and $\lambda_3 = 0$, while at $y = 0$ we have $\lambda_2 = 0$ and $\lambda_3 > 0$. Also, $\mu_1 = 0$ outside the boundary and in fact $\mu_1 \equiv 0$. Finally, μ_2 and μ_3 satisfy the same conditions as λ_2, λ_3, i.e. $\mu_2 > 0$ at $x = 0$ and $\mu_3 > 0$ at $y = 0$, and they are zero elsewhere.

From $\Delta e = (1/2m)(|p'|^2 - |p|^2) = 0$, we conclude that $\Delta p = (0, 2p'_y)$ or $\mu_3 = 2p'_y$ at $y = 0$, while $\Delta p = (2p'_x, 0)$ or $\mu_2 = 2p'_x$ at $x = 0$.

We have a periodic motion where the particle bounces back and forth between the two extreme points of the arc, with a constant angular velocity 1 or -1.

Example 4. Consider now motion of a free particle on a circular and an elliptic billiard. We begin with the circular billiard, where the constraint is given by

$$C: \qquad x^2 + y^2 \leqslant 1 \qquad \text{so } \dot{x}x + \dot{y}y \leqslant 0 \text{ if } x^2 + y^2 = 1.$$

The Lagrangian is again $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and equations (8) are now

$$\begin{array}{lllll} \dot{p}_x = \lambda x & \dot{e} - h = 0 & p_x = m\dot{x} & \Delta p_x = \mu x & \Delta e = 0 \\ \dot{p}_y = \lambda y & e = \dfrac{1}{2m}(p_x^2 + p_y^2) & p_y = m\dot{y} & \Delta p_y = \mu y & \end{array} \tag{22}$$

where now $\lambda \leqslant 0$ and $\mu \leqslant 0$ are the Lagrange multipliers, which are in fact null, unless we have a collision with the wall. In this case $\lambda(x, y)$ is the wall force acting on the particle at the instant of collision. We notice that $\Delta p = \mu(x, y)$ is the change in the momentum at the instant of collision, normal to the wall. The power of the wall force is given by

$$h = \dot{e} = \frac{1}{2m}(p_x \dot{p}_x + p_y \dot{p}_y) = \lambda(x\dot{x} + y\dot{y}) \geqslant 0$$

and is zero unless a collision with the wall is performed.

It is clear that if θ is the angle on the circle between two consecutive collisions of the particle, this angle will be repeated by symmetry as in figure 1. So, the set of consecutive collisions will be finite or dense in $S^1 = \{x^2 + y^2 = 1\}$, according to whether θ/π is rational or irrational. In the first case we have a periodic orbit, while in the second case the orbit is dense on an annulus.

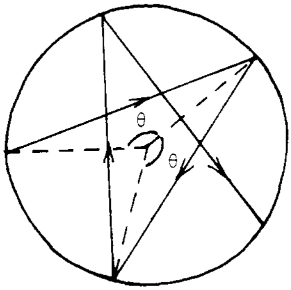

Figure 1

In fact, this can be considered as a completely integrable geodesic flow, since any elliptic billiard is the limit of the geodesic flow of oblate ellipsoids (completely integrable systems themselves) where two semi-axes are fixed while the third one tends to zero.

In particular, the circular billiard is the limit of geodesics in ellipsoids of revolution, so that the above annulus is the projection of a torus. All the invariant tori in phase space project this way of course, because of the angular momentum integral for surfaces of revolution.

For the elliptic billiard the constraint is

$$C: \qquad x^2 + \alpha^2 y^2 \leqslant 1 \qquad \text{for some constant } \alpha > 1$$

so $x\dot{x} + \alpha^2 y\dot{y} \leqslant 0$ if $x^2 + \alpha^2 y^2 = 1$. The Lagrangian is the same as above, and equations (22) are almost the same, except for the equations for $\dot{p}_y$ and Δp_y, which become

$$\dot{p}_y = \alpha^2 \lambda y \qquad \Delta p_y = \alpha^2 \mu y$$

for λ, $\mu \leqslant 0$. The power of the wall force is now given by

$$h = \dot{e} = \lambda(x\dot{x} + \alpha^2 y\dot{y}) \geqslant 0.$$

We can see that the simplest periodic orbits are those corresponding to reflections along the principal axes. A less trivial periodic orbit when $\alpha > \sqrt{2}$, can be constructed by starting with vertical trajectories $x^2 = \varkappa$ for $0 < \varkappa < 1$, and adjusting $\varkappa$. So we get an orbit symmetric with respect to the x and y axes as in figure 2.

A non-periodic orbit which for $t \to \pm\infty$ tends to the periodic orbit along the x axis can be constructed by using the property of the ellipse that reflection of an orbit from a focus always passes through the other focus. This orbit is symmetrical with respect to the x axis, and it is shown in figure 3.

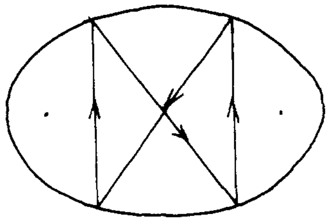

Figure 2

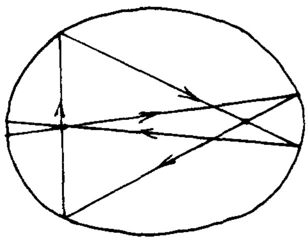

Figure 3

The elliptic billiard can also be considered as a completely integrable geodesic flow, and the projection of tori in phase space always omits some open subset which contains the foci in the elliptical region. Hence, there are no orbits dense everywhere (Arnol'd 1983). However, by considering different convex regions, we can in fact get billiards with ergodic flows (Benettin and Strelcyn 1978).

References

Arnol'd V I 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* (Berlin: Springer)

Benettin G and Strelcyn J M 1978 Numerical experiments on the free motion of a point mass moving on a plane convex region: Stochastic transition and entropy *Phys. Rev.* A **17** 773-85

Tulczyjew W M 1986 Differential geometry of mechanical systems with constraints *Atti Academia Scienze Torino* **120** 211-6